Problem 2.23

Find the potential a distance s from an infinitely long straight wire that carries a uniform line charge λ. Compute the gradient of your potential, and check that it yields the correct field.

Solution

An electrostatic field must satisfy $\nabla \times \mathbf{E} = \mathbf{0}$, which implies the existence of a potential function $-V$ that satisfies

$$\mathbf{E} = \nabla(-V) = -\nabla V.$$

The minus sign is arbitrary mathematically, but physically it indicates that a positive charge in an electric field moves from high-potential regions to low-potential regions (and vice-versa for a negative charge). To solve for V, integrate both sides along a path between two points in space with position vectors, $\mathbf{a}$ and $\mathbf{b}$, and use the fundamental theorem for gradients.

$$\begin{aligned}\int_{\mathbf{a}}^{\mathbf{b}} \mathbf{E} \cdot d\mathbf{l}_0 &= -\int_{\mathbf{a}}^{\mathbf{b}} \nabla V \cdot d\mathbf{l}_0 \\ &= -[V(\mathbf{b}) - V(\mathbf{a})] \\ &= V(\mathbf{a}) - V(\mathbf{b})\end{aligned}$$

In this context $\mathbf{a}$ is the position vector for the reference point (it cannot be infinity because the wire itself extends to infinity), and $\mathbf{b}$ is the position vector $\mathbf{r}$ for the point we're interested in knowing the electric potential.

$$\int_{\mathbf{a}}^{\mathbf{r}} \mathbf{E} \cdot d\mathbf{l}_0 = V(\mathbf{a}) - V(\mathbf{r})$$

According to Problem 2.13, the electric field around a uniformly charged straight wire is

$$\mathbf{E} = \frac{\lambda}{2\pi\epsilon_0 s}\hat{\mathbf{s}}.$$

Since the electric field is radially symmetric, the path taken from $\mathbf{a}$ to $\mathbf{r}$ is a radial one and parameterized by s_0, where $s \le s_0 \le a$.

$$\int_a^s \mathbf{E}(s_0) \cdot d\mathbf{s}_0 = V(a) - V(s)$$

Solve for the potential difference $V(s) - V(a)$.

$$\begin{aligned}V(s) - V(a) &= -\int_a^s \mathbf{E}(s_0) \cdot d\mathbf{s}_0 \\ &= \int_s^a [E(s_0)\hat{\mathbf{s}}_0] \cdot (\hat{\mathbf{s}}_0\, ds_0) \\ &= \int_s^a E(s_0)\, ds_0 \\ &= \int_s^a \frac{\lambda}{2\pi\epsilon_0 s_0}\, ds_0\end{aligned}$$

Evaluate the integral.

$$\begin{aligned} V(s) - V(a) &= \frac{\lambda}{2\pi\epsilon_0} \int_s^a \frac{ds_0}{s_0} \\ &= \frac{\lambda}{2\pi\epsilon_0} \ln s_0 \Big|_s^a \\ &= \frac{\lambda}{2\pi\epsilon_0} (\ln a - \ln s) \end{aligned}$$

Therefore,

$$\boxed{V(s) - V(a) = \frac{\lambda}{2\pi\epsilon_0} \ln \frac{a}{s}.}$$

In cylindrical coordinates

$$\begin{aligned} \nabla V &= \frac{\partial V}{\partial s}\hat{\mathbf{s}} + \frac{1}{s} \overbrace{\frac{\partial V}{\partial \phi}}^{=0} \hat{\boldsymbol{\phi}} + \overbrace{\frac{\partial V}{\partial z}}^{=0} \hat{\mathbf{z}} \\ &= \frac{dV}{ds}\hat{\mathbf{s}}, \end{aligned}$$

so differentiate both sides of the potential difference with respect to s.

$$\begin{aligned} \frac{d}{ds}[V(s) - V(a)] &= \frac{d}{ds}\left(\frac{\lambda}{2\pi\epsilon_0} \ln \frac{a}{s}\right) \\ \frac{d}{ds}[V(s)] - \underbrace{\frac{d}{ds}[V(a)]}_{=0} &= \frac{\lambda}{2\pi\epsilon_0} \frac{d}{ds}\left(\ln \frac{a}{s}\right) \\ \frac{dV}{ds} &= \frac{\lambda}{2\pi\epsilon_0} \frac{d}{ds}(\ln a - \ln s) \\ &= \frac{\lambda}{2\pi\epsilon_0}\left(0 - \frac{1}{s}\right) \\ &= -\frac{\lambda}{2\pi\epsilon_0 s} \end{aligned}$$

So

$$\nabla V = -\frac{\lambda}{2\pi\epsilon_0 s}\hat{\mathbf{s}} = -\mathbf{E}$$

as expected.